shed eddies without significant change in Strouhal number even when the stream is highly turbulent. There is, therefore, experimental data to support the application of the resonance hypothesis to cylinders. However, if the resonance hypothesis for spheres were correct, transfer should be maximum at about the same value of $L/D$ as for cylinders. Since this effect is now shown by the data reported by Raithby and Eckert, it would appear that for spheres the resonance hypothesis must be rejected. Unfortunately, no spectral data for the free-stream turbulence are available for more realistic estimates of the energy-containing frequencies. More detailed turbulence measurements in the near wake of spheres are required before this uncertainty is finally resolved.

## ACKNOWLEDGEMENTS

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# MASS TRANSFER WITH CHEMICAL REACTION IN A FINITE FALLING FILM

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## NOMENCLATURE

| | |
|---|---|
| $c$, | molar concentration in the liquid; |
| $C$, | dimensionless molar concentration; |
| $D$, | molecular diffusivity; |
| $k$, | specific reaction rate; |
| $m, n$, | order of reaction with respect to components A, B; |
| $M$, | dimensionless time, $\pi\theta/2(n + 1)$; |
| $\bar{N}_{A0}$, | average molar flux into liquid film; |
| $\bar{N}^*_{A0}$, | average molar flux for physical absorption alone; |
| $r$, | rate of reaction; |
| $u$, | liquid velocity; |
| $u_{max}$, | liquid velocity at interface; |
| $U$, | dimensionless liquid velocity, $u/u_{max}$; |
| $x, y$, | coordinates, Fig. 1; |
| $\alpha$, | $c_{B0}D_B/\nu c_{Ai}D_A$; |
| $\beta$, | $c_{B0}/\nu c_{Ai}$; |
| $\gamma$, | angle of inclination to horizontal; |
| $\delta$, | liquid film thickness; |
| $\Delta$, | dimensionless film thickness, $\delta(kc^m_{B0}c^{n-1}_{Ai}/D_A)^{\frac{1}{2}}$; |
| $\theta$, | $kc^m_{B0}c^{n-1}_{Ai}\,(x/u_{max})$; |
| $\mu$, | liquid viscosity; |
| $\nu$, | stoichiometric coefficient; |
| $\rho$, | liquid density; |
| $\phi$, | $\bar{N}_{A0}/\bar{N}^*_{A0}$. |

Subscripts

| | |
|---|---|
| A, B, | components A, B; |
| 0, | evaluated at time zero; |
| $i$, | evaluated at the interface. |

## INTRODUCTION

The penetration theory of Higbie [1] has been widely applied to unsteady state diffusional problems, with and without chemical reaction. A comprehensive survey of the literature has recently been given by Secor and Beutler [2]. As far as we can ascertain, all the solutions with chemical reaction have been obtained for the case of a semi-infinite body of liquid, although physical absorption into a finite film has been considered [13].

If the liquid were in the form of a finite falling film which is possibly a more realistic situation in chemical engineering, then the rate of mass transfer would be expected to be affected by the finite thickness of the film, since the film would become saturated after a certain distance and diffusion would cease. In this note we consider the problem of diffusion with a generalized chemical reaction into a falling film of finite thickness. The results are compared with those of Brian *et al.* [2, 3] for the semi-infinite case.

## THEORY

The physical system is shown in Fig. 1. The interface is taken to be flat, although in practice waves might form if the film is long. The velocity of the liquid is given by the Nusselt relation:

$$u = u_{\max}(1 - (y/\delta)^2) \qquad (1)$$

$$u_{\max} = \frac{\delta^2 \rho g \sin\gamma}{2\mu}. \qquad (2)$$

The chemical reaction, which is considered irreversible, takes place between a gaseous species A and a second species B which has already dissolved in the liquid phase.

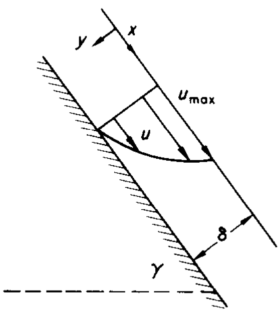

FIG. 1. The coordinate system.

The rate of consumption of A per unit volume of liquid is

$$r_A = kc_A^n c_B^m \qquad (3)$$

where $k$ is the specific reaction rate, $c_A$ and $c_B$ are instantaneous concentrations of A and B in the liquid and $m$, $n$ are the orders of reaction with respect to components A, B.

We assume that the gas mass transfer resistance is negligible and that there is no interfacial resistance. The concentrations of A in the gas and liquid phases at the interface are in equilibrium.

Ignoring diffusion in the direction of flow, the differential equations for mass transfer in the liquid are:

$$D_A \frac{\partial^2 c_A}{\partial y^2} - u\frac{\partial c_A}{\partial x} = kc_A^n c_B^m \qquad (4)$$

$$D_B \frac{\partial^2 c_B}{\partial y^2} - u\frac{\partial c_B}{\partial x} = k\nu\, c_A^n c_B^m \qquad (5)$$

where $\nu$ is the stoichiometric coefficient in A + $\nu$B → products.

It is convenient to introduce dimensionless variables as follows

$$\Delta = \delta(kc_{B0}^m\, c_{Ai}^{n-1}/D_A)^{\frac{1}{2}} \qquad (6)$$

$$\theta = kc_{B0}^m\, c_{Ai}^{n-1}\,(x/u_{\max}) \qquad (7)$$

$$C_A = c_A/c_{Ai}; \qquad C_B = c_B/c_{B0} \qquad (8)$$

$$Y = y/\delta; \qquad U = u/u_{\max} \qquad (9, 10)$$

$$\alpha = c_{B0}D_B/\nu c_{Ai}D_A. \qquad (11)$$

Equations (4) and (5) become

$$\frac{\partial^2 C_A}{\partial Y^2} - U\Delta^2 \frac{\partial C_A}{\partial x} = \Delta^2 C_A^n C_B^m \qquad (12)$$

$$\alpha \frac{\partial^2 C_B}{\partial Y^2} - \alpha U\Delta^2 \frac{\partial C_B}{\partial \theta} = \Delta^2 C_A^n C_B^m. \qquad (13)$$

These are now to be solved with the initial and boundary conditions:

$$\theta = 0,\quad 0 < Y < 1:\quad C_A = 0,\quad C_B = 1 \qquad (14)$$

$$Y = 0,\quad \theta \geqslant 0:\quad C_A = 1,\quad \partial C_B/\partial Y = 0 \qquad (15)$$

$$Y = 1,\quad \theta \geqslant 0:\quad \partial C_A/\partial Y = 0,\quad \partial C_B/\partial Y = 0. \qquad (16)$$

A material balance over an element of fluid of thickness d$y$ length d$x$ gives the mean rate of absorption $\overline{N}_{A0}$ of A,

$$\overline{N}_{A0} = \frac{1}{x}\int_0^x N_{A0}\,dx = \frac{1}{x}\int_0^\delta [C_A + (1 - C_B)\nu]\,u\,dy \qquad (17)$$

while the similar expression for physical absorption into a semi-infinite liquid is

$$\overline{N}^*_{A0} = 2\left(\frac{D_A u_{\max}}{\pi x}\right)^{\frac{1}{2}} C_{Ai} \qquad (18)$$

where $x$ in this case refers to the physical absorption. The results of our computations are expressed in terms of $\overline{N}_{A0}/\overline{N}_{A0}x$:

$$\phi = \frac{\overline{N}_{A0}}{\overline{N}^*_{A0}} = \left(\frac{\pi\Delta^2}{4\theta}\right)^{\frac{1}{2}} \int_0^1 [C_A + (1 - C_B)\beta]\,U\,dY \qquad (19)$$

where $\beta = c_{B0}/\nu c_{Ai}$. This enables easy comparison with previous work on stagnant fluids.

## COMPUTATION

The parabolic equations (12) and (13) were solved using the Crank–Nicolson method [5–8] and the resulting nonlinear equations were linearized by the method of Douglas [8]. The two resulting sets of simultaneous equations were then solved (with the initial and boundary conditions) with the aid of an IBM 7090 digital computer. The integration needed for $\phi$ was carried out using Simpson's Rule. Standard convergence and stability tests were used on every run [9], and comparison was also made with previous results [3, 4, 10]. On the basis of these tests and comparisons, our results can be considered accurate to ±3 per cent.

## RESULTS

The results are presented in Figs. 2–4 for $D_A = D_B$, $\alpha = \beta = 2$ with $n = m$ for $n = 1$–3.

The parameter $M^{\frac{1}{2}}$ [where $M = \pi\theta/2(n + 1)$] has been used as a modified dimensionless time in order to facilitate comparison with the results for a stagnant liquid [2, 3] where $\Delta \rightarrow \infty$.

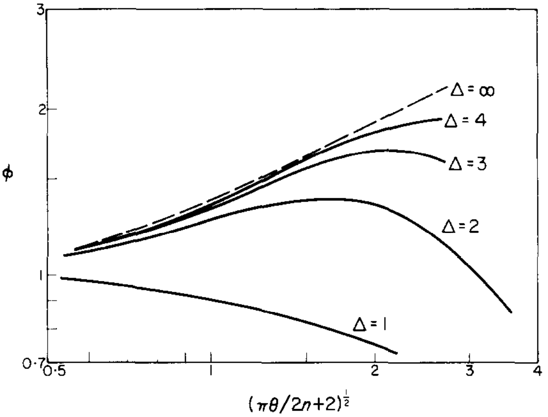

FIG. 2. Ratio of finite film mass transfer rate to transfer rate for semi-infinite film, as a function of dimensionless time. $n = m = 1$. ----- Brian *et al.* [3, 4].

Keeping $\Delta$ constant, we see that with increasing $M$ (i.e. increasing residence time) $\phi$ generally increases up to a point when the finite film thickness becomes important. Then $\phi$ decreases again and as $M^{\frac{1}{2}} \rightarrow \infty$, $\phi \rightarrow 0$. Obviously, the thinner the film, the quicker it becomes depleted in species B and the more rapidly the absorption rate approaches zero.

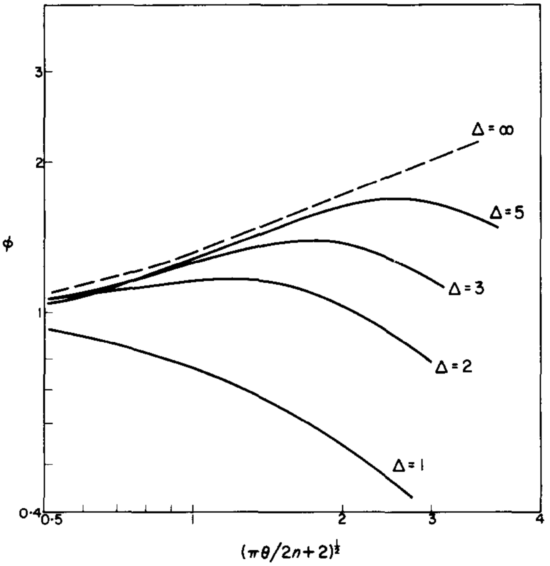

FIG. 3. Ratio of finite film mass transfer rate to transfer rate for semi-infinite film, as a function of dimensionless time; $n = m = 2$. ----- Brian *et al.* [3, 4].

As the order of reaction is increased, the film thickness has more effect. This is to be expected since the greater the order, the more rapid the reaction rate.

Although the effect of finite film thickness is quite marked in Figs. 2–4, the practical significance is of course that for thick films and short contact times, the semi-infinite theory of Brian *et al.* [3, 4] is quite adequate.

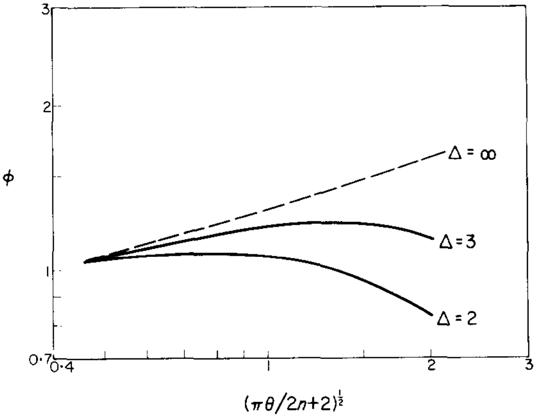

FIG. 4. Ratio of finite film mass transfer rate to transfer rate for semi-infinite film, as a function of dimensionless time; $n = m = 3$. ----- Brian *et al.* [3, 4].

For example, take a packed absorption column in which the liquid flows over Raschig rings of $\frac{1}{2}$ in. dia. operating with a typical liquid rate of 4000 lb/hft$^2$. A typical value of the film thickness for a liquid of viscosity 1 cp would be $10^{-2}$ cm [11, 12]. Thus if we put $n = m = 2$, and the path length at 1 cm

$$M^{\frac{1}{2}} = \Delta(\pi\theta/6)^{\frac{1}{2}}$$

and $\theta \simeq 0{\cdot}02$. Thus $M^{\frac{1}{2}} \simeq 0{\cdot}01\,\Delta$ and we see that from Fig. 3, the value of $\phi$ is virtually that given by the semi-infinite theory no matter what value of $\Delta$ is chosen.

If the contact length were 100 cm, which could occur in a wetted wall column such as is used to obtain absorption data, the finite film thickness would definitely have an effect on the mass transfer rate, especially for higher order reactions.

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# CONFORMAL MAPPING FOR HEAT CONDUCTION IN A REGION WITH AN UNKNOWN BOUNDARY

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## INTRODUCTION

For heat conduction in a fixed region the specification of the local surface temperature or local heat flux normal to the surface is sufficient to determine the temperature distribution within the region. The use of conformal mapping for this type of heat conduction problem in two dimensions has been discussed in [1]. If the surface temperature and heat flux are both specified the shape of the region must be free to adjust to accommodate both of these conditions. This note deals with the application of conformal mapping to two-dimensional heat conduction problems where the shape of the conducting region is unknown and will either adjust itself or is to be shaped in order to satisfy the imposed thermal conditions.

The method is best illustrated by considering a specific example. Thus, consider the geometry shown in Fig. 1. A cooled surface maintained at the temperature $t_w$ is insulated at its sides. and the length normal to the plane of the figure is sufficiently long so that the geometry can be considered two dimensional. There is a region of conducting material on the plate. The upper surface of this region is isothermal and is subjected to a unidirectional source of thermal radiation. This region might be, for example, a steady state frost layer which has formed on a very cold plate exposed to the sun's rays. Since the frost surface is at the freezing or sublimation temperature consistent with the surrounding conditions, it will be at a constant temperature $t_s$. It is desired to find the shape that the frost region assumes and the heat flow through this region since this determines how well the frost layer insulates the surface. Alternatively the results can be interpreted as the solution to the problem of finding the shape of a conducting region that will provide a uniform temperature $t_s$ at its surface when this surface is subjected to incident radiation. It will be assumed that $t_s$ is sufficiently low so that radiation emitted from the surface can be neglected compared with the absorbed incident radiation.

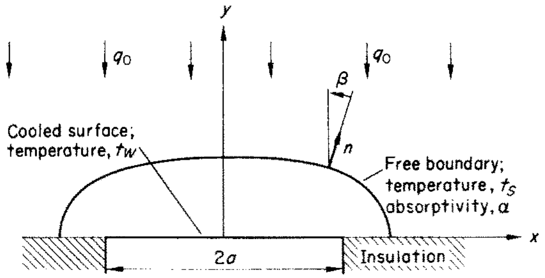

Fig. 1. Cross section of two-dimensional region with free boundary at uniform temperature $t_s$ and with uniform absorptivity $\alpha$ exposed to unidirectional radiation $q_0$.